

Non degeneracy for solutions of singularly perturbed nonlinear elliptic problems on symmetric Riemannian manifolds

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January 27, 2013

Abstract

Given a symmetric Riemannian manifold (M, g) , we show some results of genericity for non degenerate sign changing solutions of singularly perturbed nonlinear elliptic problems with respect to the parameters: the positive number ε and the symmetric metric g . Using these results we obtain a lower bound on the number of non degenerate solutions which change sign exactly once.

Keywords: symmetric Riemannian manifolds, non degenerate sign changing solutions, singularly perturbed nonlinear elliptic problems

AMS subject classification: 58G03, 58E30

1 Introduction

Let (M, g) be a smooth connected compact Riemannian manifold of finite dimension $n \geq 2$ embedded in \mathbb{R}^N . Let us consider the problem

$$\begin{cases} -\varepsilon^2 \Delta_g u + u = |u|^{p-2}u & \text{in } M \\ u \in H_g^1(M) \end{cases} \quad (1)$$

Recently there have been some results on the influence of the topology (see [3, 12, 23]) and the geometry (see [5, 7, 16]) of M on the number of positive solutions of problem (1). This problem has similar features with the Neumann problem on a flat domain, which has been largely studied in literature (see [6, 8, 10, 11, 13, 18, 19, 24, 25, 26]).

Concerning the sign changing solution the first result is contained in [15] where it is showed the existence of a solution with one positive peak and one negative peak when the scalar curvature of (M, g) is non constant.

Moreover in [9] the authors give a multiplicity result for solutions which change sign exactly once when the Riemannian manifold is symmetric with respect to an orthogonal involution τ using the equivariant Ljusternik Schnirelmann category.

In this paper we are interested in studying the non degeneracy of changing sign solutions when the Riemannian manifold (M, g) is symmetric.

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We consider the problem

$$\begin{cases} -\varepsilon^2 \Delta_g u + u = |u|^{p-2} u & u \in H_g^1(M) \\ u(\tau x) = -u(x) & \forall x \in M \end{cases} \quad (2)$$

where $\tau : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is an orthogonal linear transformation such that $\tau \neq \text{Id}$, $\tau^2 = \text{Id}$ (Id being the identity on \mathbb{R}^N). Here the compact connected Riemannian manifold (M, g) of dimension $n \geq 2$ is a regular submanifold of \mathbb{R}^N invariant with respect to τ . Let $M_\tau = \{x \in M : \tau x = x\}$. In the case $M_\tau \neq \emptyset$ we assume that M_τ is a regular submanifold of M . In the following $H_g^\tau = \{u \in H_g^1(M) : \tau^* u = u\}$ where the linear operator $\tau^* : H_g^1 \rightarrow H_g^1$ is $\tau^* u = -u(\tau(x))$.

We obtain the following genericity results about the non degeneracy of changing sign solutions of (2) with respect to the parameters: the positive number ε , and the symmetric metric g (i.e. $g(\tau x) = g(x)$).

Theorem 1. *Given $g_0 \in \mathcal{M}^k$, the set*

$$D = \left\{ (\varepsilon, h) \in (0, 1) \times \mathcal{B}_\rho : \begin{array}{l} \text{any } u \in H_{g_0}^\tau \text{ solution of} \\ -\varepsilon^2 \Delta_{g_0+h} u + u = |u|^{p-2} u \text{ is not degenerate} \end{array} \right\}$$

is a residual subset of $(0, 1) \times \mathcal{B}_\rho$.

Remark 2. By the previous result we prove that, given $g_0 \in \mathcal{M}^k$ and $\varepsilon_0 > 0$, the set

$$D^* = \left\{ h \in \mathcal{B}_\rho : \begin{array}{l} \text{any } u \in H_{g_0}^\tau \text{ solution of} \\ -\varepsilon^2 \Delta_{g_0+h} u + u = |u|^{p-2} u \text{ is not degenerate} \end{array} \right\}$$

is a residual subset of \mathcal{B}_ρ .

In the following we set

$$m_{\varepsilon_0, g_0}^\tau = \inf_{u \in \mathcal{N}_{\varepsilon_0, g_0}^\tau} J_{\varepsilon_0, g_0}(u)$$

where

$$\begin{aligned} J_{\varepsilon_0, g_0}(u) &= \frac{1}{\varepsilon_0^n} \int_M \left[\frac{1}{2} (\varepsilon_0^2 |\nabla_g u|^2 + u^2) - \frac{1}{p} |u|^p \right] d\mu_{g_0} \\ \mathcal{N}_{\varepsilon_0, g_0}^\tau &= \{u \in H_{g_0}^\tau(M) \setminus \{0\} : J'_{\varepsilon_0, g_0}(u)[u] = 0\}. \end{aligned}$$

Theorem 3. *Given $g_0 \in \mathcal{M}^k$ and $\varepsilon_0 > 0$. If there exists $\mu > m_{\varepsilon_0, g_0}^\tau$ which is not a critical level of the functional $J_{\varepsilon_0, g_0}^\tau$, then the set*

$$D^\dagger = \left\{ h \in \mathcal{B}_\rho : \begin{array}{l} \text{any } u \in H_{g_0+h}^\tau \text{ solution of} \\ -\varepsilon^2 \Delta_{g_0+h} u + u = |u|^{p-2} u \text{ with } J_{\varepsilon_0, g_0}^\tau(u) < \mu \text{ is not degenerate} \end{array} \right\}$$

is an open dense subset of \mathcal{B}_ρ .

Here the set \mathcal{B}_ρ is the ball centered at 0 with radius ρ in the space \mathcal{S}^k , where ρ is small enough and \mathcal{S}^k is the Banach space of all C^k , $k \geq 3$, symmetric covariants 2-tensor $h(x)$ on M such that $h(x) = h(\tau x)$ for $x \in M$. $\mathcal{M}^k \subset \mathcal{S}^k$ is the set of all C^k Riemannian metrics g on M such that $g(x) = g(\tau x)$.

These results can be applied to obtain a lower bound for the number of non degenerate solutions of (2) which change sign exactly once when M is invariant with respect to the involution $\tau = -\text{Id}$ and $0 \notin M$. We get the following propositions.

Proposition 4. *Given $g_0 \in \mathcal{M}^k$, the set*

$$\mathcal{A} = \left\{ \begin{array}{l} (\varepsilon, h) \in (0, \bar{\varepsilon}) \times \mathcal{B}_\rho : \text{the equation } -\varepsilon^2 \Delta_{g_0+h} u + u = |u|^{p-2} u \\ \text{has at least } P_1(M/G) \text{ pairs of non degenerate solutions} \\ (u, -u) \in H_g^\tau \setminus \{0\} \text{ which change sign exactly once} \end{array} \right\}$$

is a residual subset of $(0, 1) \times \mathcal{B}_\rho$.

Proposition 5. *Given $g_0 \in \mathcal{M}^k$ and $\varepsilon_0 > 0$, if there exists $\mu > m_{\varepsilon_0, g_0}^\tau$ not a critical value of J_{ε_0, g_0} in $H_{g_0}^\tau$, then the set*

$$\mathcal{A}^\dagger = \left\{ \begin{array}{l} h \in \mathcal{B}_\rho : \text{the equation } -\varepsilon_0^2 \Delta_{g_0+h} u + u = |u|^{p-2} u \\ \text{has at least } P_1(M/G) \text{ pairs of non degenerate solutions} \\ (u, -u) \in H_g^\tau \setminus \{0\} \text{ which change sign exactly once} \end{array} \right\}$$

is an open dense subset of \mathcal{B}_ρ .

Here $P_t(M/G)$ is the Poincaré polynomial of the manifold M/G , where $G = \{\text{Id}, -\text{Id}\}$, and $P_1(M/G)$ is when $t = 1$. By definition we have $P_t(M/G) = \sum_k \dim H_k(M/G) \cdot t^k$ where $H_k(M/G)$ is the k -th homology group with coefficients in some field.

The paper is organized as follows. In Section 2 we recall some preliminary results. In Section 3 we sketch the proof of the results of genericity (theorems 1 and 3) using some technical lemmas proved in Section 4. In Section 5 we prove propositions 4 and 5.

2 Preliminaries

Given a connected n dimensional C^∞ compact manifold M without boundary endowed with a Riemannian metric g , we define the functional spaces L_g^p , $L_{\varepsilon, g}^p$, H_g^1 and $H_{\varepsilon, g}^1$, for $2 \leq p < 2^*$ and a given $\varepsilon \in (0, 1)$. The inner products on L_g^2 and H_g^1 are, respectively

$$\langle u, v \rangle_{L_g^2} = \int_M u v d\mu_g \quad \langle u, v \rangle_{H_g^1} = \int_M (\nabla u \nabla v + uv) d\mu_g,$$

while the inner products on $L_{\varepsilon, g}^2$ and $H_{\varepsilon, g}^1$ are, respectively

$$\langle u, v \rangle_{L_{\varepsilon, g}^2} = \frac{1}{\varepsilon^n} \int_M u v d\mu_g \quad \langle u, v \rangle_{H_{\varepsilon, g}^1} = \frac{1}{\varepsilon^n} \int_M (\varepsilon^2 \nabla u \nabla v + uv) d\mu_g.$$

Finally, the norms in L_g^p and $L_{\varepsilon, g}^p$ are

$$\|u\|_{L_g^p}^p = \int_M |u|^p d\mu_g \quad \|u\|_{L_{\varepsilon, g}^p}^p = \frac{1}{\varepsilon^n} \int_M |u|^p d\mu_g.$$

We define also the space of symmetric L^p and H^1 functions as

$$L_g^{p, \tau} = \{u \in L_g^p(M) : \tau^* u = u\} \quad H_g^\tau = \{u \in H_g^1(M) : \tau^* u = u\}$$

As defined in the introduction, \mathcal{S}^k is the space of all C^k symmetric covariants 2-tensor $h(x)$ on M such that $h(x) = h(\tau x)$ for $x \in M$. We define a

norm $\|\cdot\|_k$ in \mathcal{S}^k in the following way. We fix a finite covering $\{V_\alpha\}_{\alpha \in L}$ of M where (V_α, ψ_α) is an open coordinate neighborhood. If $h \in \mathcal{S}^k$, denoting h_{ij} the components of h with respect to local coordinates (x_1, \dots, x_n) on V_α , we define

$$\|h\|_k = \sum_{\alpha \in L} \sum_{|\beta| \leq k} \sum_{i,j=1}^n \sup_{\psi_\alpha(V_\alpha)} \left| \frac{\partial^\beta h_{ij}}{\partial x_1^{\beta_1} \dots \partial x_n^{\beta_n}} \right|.$$

The set \mathcal{M}^k of all C^k Riemannian metrics g on M such that $g(x) = g(\tau x)$ is an open set of \mathcal{S}^k .

Given $g_0 \in \mathcal{M}^k$ a symmetric Riemannian metric on M , we notice that there exists $\rho > 0$ (which does not depend on ε if $0 < \varepsilon < 1$) such that, if $h \in \mathcal{B}_\rho$ the sets H_{ε, g_0+h}^1 and H_{ε, g_0}^1 are the same and the two norms $\|\cdot\|_{H_{\varepsilon, g_0+h}^1}$ and $\|\cdot\|_{H_{\varepsilon, g_0}^1}$ are equivalent. The same for L_{ε, g_0+h}^2 and L_{ε, g_0}^2 . If $h \in \mathcal{B}_\rho$ and $\varepsilon \in (0, 1)$ we set

$$\begin{aligned} E_h^\varepsilon(u, v) &= \langle u, v \rangle_{H_{\varepsilon, g_0+h}^1} & \forall u, v \in H_{\varepsilon, g_0+h}^1 \\ G_h^\varepsilon(u, v) &= \langle u, v \rangle_{L_{\varepsilon, g_0+h}^2} & \forall u, v \in L_{\varepsilon, g_0+h}^2 \\ N(\varepsilon, h)(u) &= N_h^\varepsilon(u) = \|u\|_{L_{\varepsilon, g_0+h}^p}^p & \forall u \in L_{\varepsilon, g_0+h}^p \end{aligned}$$

We introduce the map A_h^ε which will be used in the following section.

Remark 6. If $h \in \mathcal{B}_\rho$ and $0 < \varepsilon < 1$, there exists a unique linear operator

$$A(\varepsilon, h) := A_h^\varepsilon : L_{g_0}^{p', \tau}(M) \rightarrow H_{g_0}^\tau$$

such that $E_h^\varepsilon(A_h^\varepsilon(u), v) = G_h^\varepsilon(u, v)$ for all $u \in L_{\varepsilon, g_0}^{p', \tau}$, $v \in H_{\varepsilon, g_0}^\tau$ with $2 \leq p < 2^*$. Moreover $E_h^\varepsilon(A_h^\varepsilon(u), v) = E_h^\varepsilon(u, A_h^\varepsilon(v))$ for $u, v \in H_{\varepsilon, g_0}^\tau$.

Also, we have that $A_h^\varepsilon = i_{\varepsilon, g_0}^*$ where i_{ε, g_0}^* is the adjoint of the compact embedding $i_{\varepsilon, g_0} : H_{\varepsilon, g_0}^\tau(M) \rightarrow L_{\varepsilon, g_0}^{p, \tau}(M)$ with $2 \leq p < 2^*$. We recall that, if $h \in \mathcal{B}_\rho$ with ρ small enough and $\varepsilon > 0$, then H_{ε, g_0}^1 and H_{ε, g_0+h}^1 (as well as L_{ε, g_0}^p and L_{ε, g_0+h}^p) are the same as sets and the norms are equivalent. This is the reason why we can define A_h^ε on $L_{g_0}^{p', \tau}$ with values in $H_{g_0}^\tau$. We summarize some technical results contained in lemmas 2.1, 2.2 and 2.3 of [14].

Lemma 7. *Let $g_0 \in \mathcal{M}^k$ and ρ small enough. We have*

1. *The map $E : (0, 1) \times \mathcal{B}_\rho \rightarrow \mathcal{L}(H_{g_0}^\tau \times H_{g_0}^\tau, \mathbb{R})$ defined by $E(\varepsilon, h) := E_h^\varepsilon$ is of class C^1 and it holds, for $u, v \in H_{g_0}^\tau(M)$ and $h \in \mathcal{S}^k$*

$$\begin{aligned} E'(\varepsilon_0, h_0)[\varepsilon, h](u, v) &= \frac{1}{2\varepsilon_0^n} \int_M \text{tr}(g^{-1}h) u v d\mu_g + \frac{1}{\varepsilon_0^{n-2}} \int_M \langle \nabla_g u, \nabla_g v \rangle_{b(h)} d\mu_g \\ &\quad - \frac{n\varepsilon}{\varepsilon_0^{n+1}} \int_M u v d\mu_g - \frac{(n-2)\varepsilon}{\varepsilon_0^{n-1}} \int_M \langle \nabla_g u, \nabla_g v \rangle d\mu_g \end{aligned}$$

with the 2-tensor $b(h) := \frac{1}{2} \text{tr}(g^{-1}h)g - g^{-1}hg^{-1}$

2. *The map $G : (0, 1) \times \mathcal{B}_\rho \rightarrow \mathcal{L}(L_{g_0}^{p', \tau}, H_{g_0}^\tau)$ defined by $G(\varepsilon, h) := G_h^\varepsilon$ is of class C^1 and it holds, for $u, v \in H_{g_0}^\tau(M)$ and $h \in \mathcal{S}^k$*

$$G'(\varepsilon_0, h_0)[\varepsilon, h](u, v) = \frac{1}{2\varepsilon_0^n} \int_M \text{tr}(g^{-1}h) u v d\mu_g - \frac{n\varepsilon}{\varepsilon_0^{n+1}} \int_M u v d\mu_g$$

3. The map $A : (0, 1) \times \mathcal{B}_\rho \rightarrow \mathcal{L}(H_{g_0}^\tau \times H_{g_0}^\tau, \mathbb{R})$ is of class C^1 and for any $u, v \in H_{g_0}^\tau(M)$ and $h \in \mathcal{S}^k$ we have

$$E'(\varepsilon_0, h_0) [\varepsilon, h] (A_{h_0}^{\varepsilon_0}(u), v) + E_{h_0}^{\varepsilon_0}(A'(\varepsilon_0, h_0) [\varepsilon, h] (u), v) = G'(\varepsilon_0, h_0) [\varepsilon, h] (u, v)$$

4. The map $N : (0, 1) \times \mathcal{B}_\rho \rightarrow C^0(H_{g_0}^\tau, \mathbb{R})$ defined by $(\varepsilon, h) \mapsto N_h^\varepsilon(\cdot)$ is of class C^1 and it holds, for $u \in H_{g_0}^\tau(M)$ and $h \in \mathcal{S}^k$

$$N'(\varepsilon_0, h_0) [\varepsilon, h] (u) = \frac{1}{2\varepsilon_0^n} \int_M \text{tr}(g^{-1}h) |u|^p d\mu_g - \frac{n\varepsilon}{\varepsilon_0^{n+1}} \int_M |u|^p d\mu_g$$

In all these formulas $g = g_0 + h_0$ with $h_0 \in \mathcal{B}_\rho$.

We recall two abstract results in transversality theory (see [20, 21, 22]) which will be fundamental for our results.

Theorem 8. Let X, Y, Z be three real Banach spaces and let $U \subset X$, $V \subset Y$ be two open subsets. Let F be a C^1 map from $V \times U$ into Z such that

- (i) For any $y \in V$, $F(y, \cdot) : x \rightarrow F(y, x)$ is a Fredholm map of index 0.
- (ii) 0 is a regular value of F , that is $F'(y_0, x_0) : Y \times X \rightarrow Z$ is onto at any point (y_0, x_0) such that $F(y_0, x_0) = 0$.
- (iii) The map $\pi \circ i : F^{-1}(0) \rightarrow Y$ is proper, where i is the canonical embedding form $F^{-1}(0)$ into $Y \times X$ and π is the projection from $Y \times X$ onto Y .

Then the set

$$\theta = \{y \in V : 0 \text{ is a regular value of } F(y, \cdot)\}$$

is a dense open subset of V

Theorem 9. If F satisfies (i) and (ii) and

- (iv) The map $\pi \circ i$ is σ -proper, that is $F^{-1}(0) = \bigcup_{s=1}^{+\infty} C_s$ where C_s is a closed set and the restriction $\pi \circ i|_{C_s}$ is proper for any s

then the set θ is a residual subset of V

3 Sketch of the proof of theorems 1 and 3.

Given $g_0 \in \mathcal{M}^k$, we introduce the map $F : (0, 1) \times \mathcal{B}_\rho \times H_{g_0}^\tau \setminus \{0\} \rightarrow H_{g_0}^\tau$ defined by

$$F(\varepsilon, h, u) = u - A_h^\varepsilon(|u|^{p-2}u).$$

By the regularity of the map A (see 3 of Lemma 7) we get the map F is of class C^1 . We are going to apply transversality Theorem 8 to the map F , in order to prove Theorem 1. In this case we have $X = H_{g_0}^\tau$, $Y = \mathbb{R} \times \mathcal{S}^k$, $Z = H_{g_0}^\tau$, $U = H_{g_0}^\tau \setminus \{0\}$ and $V = (0, 1) \times \mathcal{B}_\rho \subset \mathbb{R} \times \mathcal{S}^k$.

Assumptions (i) and (iv) are verified in Lemma 10 and in Lemma 11. Using Lemma 12 we can verify (ii).

Indeed, we have to verify that for $(\varepsilon_0, h_0, u_0) \in V \times U$ such that $F(\varepsilon_0, h_0, u_0) = 0$ and for any $b \in H_{g_0}^\tau$, there exists $(\varepsilon, h, v) \subset \mathcal{S}^k \times H_{g_0}^\tau$ such that

$$F'_u(\varepsilon_0, h_0, u_0)[v] + F'_{\varepsilon, h}(\varepsilon_0, h_0, u_0)[\varepsilon, h] = b.$$

We recall that the operator

$$v \mapsto F'_u(\varepsilon_0, h_0, u_0)[v] = v - (p-1)i_{\varepsilon_0, g_0+h}^*(|u_0|^{p-1}u_0v)$$

is selfadjoint in $H_{\varepsilon_0, g_0+h}^\tau$ and is a Fredholm operator of index 0. Then

$$\text{Im } F'_u(\varepsilon_0, h_0, u_0) \oplus \ker F'_u(\varepsilon_0, h_0, u_0) = H_{g_0}^\tau.$$

Let $\{w_1, \dots, w_\nu\}$ be a basis of $\ker F'_u(\varepsilon_0, h_0, u_0)[v]$. We consider the linear functional $f_i : \mathbb{R} \times \mathcal{S}^k \rightarrow \mathbb{R}$ defined by

$$f_i(\varepsilon, h) = (F'_{\varepsilon, h}(\varepsilon_0, h_0, u_0)[\varepsilon, h], w_i)_{H_{\varepsilon_0, g_0+h}^\tau} \quad i = 1, \dots, \nu.$$

By Lemma 12 we get that the linear functionals f_i are independent. Therefore assumption (ii) is verified. At this point by transversality theorems we get that the set

$$\left\{ (\varepsilon, h) \in (0, 1) \times \mathcal{B}_\rho : \begin{array}{l} \text{any } u \in H_{g_0}^\tau \setminus \{0\} \text{ solution of} \\ -\varepsilon^2 \Delta_{g_0+h} u + u = |u|^{p-2} u \text{ is not degenerate} \end{array} \right\}$$

is a residual subset of $(0, 1) \times \mathcal{B}_\rho$. On the other hand we observe that 0 is a non degenerate solution of $-\varepsilon^2 \Delta_{g_0+h} u + u = |u|^{p-2} u$, for any $\varepsilon > 0$ and any $h \in \mathcal{B}_\rho$. Then, we complete the proof of Theorem 1.

The proof of Remark 2 is analog to the proof of Theorem 1 using Corollary 13.

We now formulate the problem for Theorem 3. Given $g_0 \in \mathcal{M}^k$ and $\varepsilon_0 > 0$, we assume that there exists $\mu > m_{\varepsilon_0, g_0}^\tau$ which is not a critical level for the functional J_{ε_0, g_0} . It is clear that any $\mu_0 \in (0, m_{\varepsilon_0, g_0}^\tau)$ is not a critical value of J_{ε_0, g_0} . We set

$$\mathcal{D} = \{u \in H_{g_0}^\tau : \mu_0 < J_{\varepsilon_0, g_0}(u) < \mu\}.$$

Now we introduce the C^1 map $H : \mathcal{B}_\rho \times \mathcal{A} \rightarrow H_{g_0}^1$ defined by

$$H(h, u) = u - A_h^{\varepsilon_0}(|u|^{p-2}u) = F(\varepsilon_0, h, u). \quad (3)$$

We are going to apply transversality theorem 9 to the map H . In this case $X = H_{g_0}^\tau$, $Y = \mathcal{S}^k$, $Z = H_{g_0}^1(M)$, $U = \mathcal{D} \subset H_{g_0}^\tau$ and $V = \mathcal{B}_\rho \subset \mathcal{S}^k$. It is easy to verify assumptions (i) and (ii) for the map H using Lemma 10, Lemma 12 and Corollary 13. Using Lemma 14 we can verify assumption (iii) so we are in position to apply Theorem 9 and to get the following statement: the set

$$\left\{ h \in \mathcal{B}_\rho : \begin{array}{l} \text{any } u \in H_{g_0}^\tau \text{ solution of } -\varepsilon_0^2 \Delta_{g_0+h} u + u = |u|^{p-2} u \\ \text{such that } \mu_0 < J_{\varepsilon_0, g_0}(u) < \mu \text{ is not degenerate} \end{array} \right\}$$

is an open dense subset of \mathcal{B}_ρ . Nevertheless 0 is a non degenerate solution of $-\varepsilon_0^2 \Delta_{g_0+h} u + u = |u|^{p-2} u$ for any h , and there is no solution $u \neq 0$ with $J_{\varepsilon_0, g_0}(u) < \mu_0$, so we get the claim.

4 Technical lemmas

In this section we show some lemmas in order to complete the proof of the results of genericity of non degenerate critical points.

Lemma 10. *For any $(\varepsilon, h) \in (0, 1) \times \mathcal{B}_\rho$ the map $u \mapsto F(\varepsilon, h, u)$ with $u \in H_{g_0}^\tau$ is a Fredholm map of index zero.*

Proof. By the definition of the map A , we have

$$F'(\varepsilon_0, h_0, u_0)[v] = v - (p-1)A_{h_0}^{\varepsilon_0}[|u_0|^{p-2}v] = v - Kv,$$

where $K(v) := (p-1)i_{\varepsilon_0, g_0+h_0}^*[|u_0|^{p-2}v]$. We will verify that $K : H_{\varepsilon_0, g_0}^\tau \rightarrow H_{\varepsilon_0, g_0}^\tau$ is compact. Thus $K : H_{g_0}^\tau \rightarrow H_{g_0}^\tau$ is compact and the claim follows. In fact, in v_n is bounded in $H_{g_0}^\tau$, v_n is also bounded in $H_{\varepsilon_0, g_0+h_0}^\tau$ because $h_0 \in \mathcal{B}_\rho$. Then, up to subsequence, v_n converges to v in $L_{\varepsilon_0, g_0+h_0}^t$ for $2 \leq t < 2^*$. So we have

$$\int_M ||u_0|^{p-2}(v_n - v)|^{p'} d\mu_g \leq \left(\int_M |u_0|^p d\mu_g \right)^{\frac{p-2}{p-1}} \left(\int_M |v_n - v|^p d\mu_g \right)^{\frac{1}{p-1}} \rightarrow 0.$$

Therefore $i_{\varepsilon_0, g_0+h_0}^*[|u_0|^{p-2}(v_n - v)] \rightarrow 0$ in $H_{\varepsilon_0, g_0+h_0}^\tau$ and also in $H_{\varepsilon_0, g_0}^\tau$. \square

Lemma 11. *The map $\pi \circ i : F^{-1}(0) \rightarrow \mathbb{R} \times \mathcal{S}^k$ is σ -proper. Here i is the canonical immersion from $F^{-1}(0)$ into $\mathbb{R} \times \mathcal{S}^k \times H_{g_0}^\tau$ and π is the projection from $\mathbb{R} \times \mathcal{S}^k \times H_{g_0}^\tau$ into $\mathbb{R} \times \mathcal{S}^k$.*

Proof. Set $I_{g_0}(u, R)$ the open ball in $H_{g_0}^\tau$ centered in u with radius R . We have $F^{-1}(0) = \cup_{s=1}^{+\infty} C_s$ where

$$C_s = \left\{ \left[\frac{1}{s}, 1 - \frac{1}{s} \right] \times \overline{\mathcal{B}_{\rho - \frac{1}{s}}} \times \left\{ \overline{I_{g_0}(0, s)} \setminus I_{g_0}\left(0, \frac{1}{s}\right) \right\} \right\} \cap F^{-1}(0).$$

We had to prove that $\pi \circ i : C_s \rightarrow \mathbb{R} \times \mathcal{S}^k$ is proper, that is if $h_n \rightarrow h_0$ in $\overline{\mathcal{B}_{\rho - \frac{1}{s}}}$, $\varepsilon_n \rightarrow \varepsilon_0$ in $[\frac{1}{s}, 1 - \frac{1}{s}]$, $u_n \in \left\{ \overline{I_{g_0}(0, s)} \setminus I_{g_0}\left(0, \frac{1}{s}\right) \right\}$, and $F(\varepsilon_n, h_n, u_n) = 0$, then, up to a subsequence, the sequence $\{u_n\}$ converges to $u_0 \in \left\{ \overline{I_{g_0}(0, s)} \setminus I_{g_0}\left(0, \frac{1}{s}\right) \right\}$. Since $\{u_n\}$ is bounded in $H_{g_0}^1$, then it is bounded in $H_{g_0+h_0}^1$, since the two spaces are equivalent because $h_0 \in \mathcal{B}_\rho$. Thus u_n converges, up to subsequence, to u_0 in $L_{g_0+h_0}^p$ and in $L_{\varepsilon_0, g_0+h_0}^p$ for $2 \leq p < 2^*$, so $|u_n|^{p-2}u_n \rightarrow |u_0|^{p-2}u_0$ in $L_{\varepsilon_0, g_0+h_0}^{p'}$ and, by continuity of $A_{h_0}^{\varepsilon_0}$,

$$i_{\varepsilon_0, g_0+h_0}^*(|u_n|^{p-2}u_n) = A_{h_0}^{\varepsilon_0}(|u_n|^{p-2}u_n) \rightarrow A_{h_0}^{\varepsilon_0}(|u_0|^{p-2}u_0) \text{ in } H_{\varepsilon_0, g_0+h_0}^1 = H_{\varepsilon_0, g_0}^1. \quad (4)$$

By the regularity of the map A we have, for some $\theta \in (0, 1)$

$$\begin{aligned} \|A_{h_n}^{\varepsilon_n}(|u_n|^{p-2}u_n) - A_{h_0}^{\varepsilon_0}(|u_n|^{p-2}u_n)\|_{H_{\varepsilon_0, g_0}^1} &\leq \|u_n\|_{L_{\varepsilon_0, g_0}^{p'}}^{p-1} [|\varepsilon_n - \varepsilon_0| + \|h_n - h_0\|_k] \times \\ &\quad \times \|A'(\varepsilon_0 + \theta(\varepsilon_n - \varepsilon_0), h_0 + \theta(h_n - h_0))\|_{\mathcal{L}((0,1) \times \mathcal{B}_\rho, \mathcal{L}(L_{\varepsilon_0, g_0}^{p'}, H_{\varepsilon_0, g_0}^1))}. \end{aligned} \quad (5)$$

By (4) and (5) we get that $A_{h_n}^{\varepsilon_n}(|u_n|^{p-2}u_n) \rightarrow A_{h_0}^{\varepsilon_0}(|u_0|^{p-2}u_0)$ in H_{ε_0, g_0}^1 then in $H_{g_0}^\tau$. Since

$$0 = F(\varepsilon_n, h_n, u_n) = u_n - A_{h_n}^{\varepsilon_n}(|u_n|^{p-2}u_n)$$

we get the claim. \square

Lemma 12. For any $(\varepsilon_0, h_0, u_0) \in (0, 1) \times \mathcal{B}_\rho \times H_{g_0}^\tau \setminus \{0\}$ such that $F(\varepsilon_0, h_0, u_0) = 0$, it holds that, if $w \in \ker F'_u(\varepsilon_0, h_0, u_0)$ and

$$\langle F'_{\varepsilon, h}(\varepsilon_0, h_0, u_0) [\varepsilon, h], w \rangle_{H_{\varepsilon_0, g_0 + h_0}^1} = 0 \quad \forall \varepsilon \in \mathbb{R}, \quad h \in \mathcal{S}^k,$$

then $w = 0$.

Proof. Step 1. By the definition of F and Lemma 7 we get

$$F'_{\varepsilon, h}(\varepsilon_0, h_0, u_0) [\varepsilon, h] = -A'(\varepsilon_0, h_0) [\varepsilon, h] (|u_0|^{p-2} u_0) \quad (6)$$

and so

$$\begin{aligned} & \langle F'_{\varepsilon, h}(\varepsilon_0, h_0, u_0) [\varepsilon, h], w \rangle_{H_{g_0 + h_0, \varepsilon_0}^1} \\ &= -E_{h_0}^{\varepsilon_0} (A'(\varepsilon_0, h_0) [\varepsilon, h] (|u_0|^{p-2} u_0), w) = \\ &= -G'(\varepsilon_0, h_0) [\varepsilon, h] (|u_0|^{p-2} u_0, w) + E'(\varepsilon_0, h_0) [\varepsilon, h] (u_0, w) = \\ &= -\frac{1}{2\varepsilon_0^n} \int_M \text{tr}(g^{-1}h) |u_0|^{p-2} u_0 w d\mu_g + \frac{n\varepsilon}{\varepsilon_0^{n+1}} \int_M |u_0|^{p-2} u_0 w d\mu_g \\ &\quad + \frac{1}{2\varepsilon_0^n} \int_M \text{tr}(g^{-1}h) u_0 w d\mu_g + \frac{1}{\varepsilon_0^{n-2}} \int_M \langle \nabla_g u_0, \nabla_g w \rangle_{b(h)} d\mu_g \\ &\quad - \frac{n\varepsilon}{\varepsilon_0^{n+1}} \int_M u_0 w d\mu_g - \frac{(n-2)\varepsilon}{\varepsilon_0^{n-1}} \int_M \langle \nabla_g u_0, \nabla_g w \rangle d\mu_g. \end{aligned}$$

Here we use that $A_{h_0}^{\varepsilon_0}(|u_0|^{p-2} u_0) = u_0$. Moreover $g = g_0 + h_0$ with $h_0 \in \mathcal{B}_\rho$ and $b(h) := \frac{1}{2} \text{tr}(g^{-1}h)g - g^{-1}hg^{-1}$.

If we choose $\varepsilon = 0$, by the previous equation we get

$$\begin{aligned} \langle F'_{\varepsilon, h}(\varepsilon_0, h_0, u_0) [0, h], w \rangle_{H_{\varepsilon_0, g_0 + h_0}^1} &= \frac{1}{2\varepsilon_0^n} \int_M \text{tr}(g^{-1}h) [u_0 - |u_0|^{p-2} u_0] w d\mu_g + \\ &\quad + \frac{1}{\varepsilon_0^{n-2}} \int_M \langle \nabla_g u_0, \nabla_g w \rangle_{b(h)} d\mu_g \quad (7) \end{aligned}$$

Step 2. We prove that, if $\langle F'_{\varepsilon, h}(\varepsilon_0, h_0, u_0) [0, h], w \rangle_{H_{\varepsilon_0, g_0 + h_0}^1} = 0 \quad \forall h \in \mathcal{S}^k$, then it holds

$$\langle \nabla_g u_0(\xi), \nabla_g w(\xi) \rangle_{b(h)} = 0 \quad \text{for all } \xi \in M.$$

Given $\xi_0 \in M$, we consider the normal coordinates at ξ_0 and we set

$$\tilde{u}_0(x) = u_0(\exp_{\xi_0} x), \quad \tilde{w}(x) = w(\exp_{\xi_0} x), \quad \text{for } x \in B(0, R) \subset \mathbb{R}^n.$$

We will prove that $\frac{\partial \tilde{u}_0(0)}{\partial x_1} \frac{\partial \tilde{w}(0)}{\partial x_2} + \frac{\partial \tilde{u}_0(0)}{\partial x_2} \frac{\partial \tilde{w}(0)}{\partial x_1} = 0$. Analogously we can get $\frac{\partial \tilde{u}_0(0)}{\partial x_i} \frac{\partial \tilde{w}(0)}{\partial x_j} + \frac{\partial \tilde{u}_0(0)}{\partial x_j} \frac{\partial \tilde{w}(0)}{\partial x_i} = 0$.

If $\xi_0 \neq \tau \xi_0$, we assume that $B_g(\xi_0, R) \cap B_g(\tau \xi_0, R) = \emptyset$. Then choosing $h \in \mathcal{S}^k$ vanishing outside $B_g(\xi_0, R) \cup B_g(\tau \xi_0, R)$, by the fact that $h(\tau x) = h(x)$ on M , by (17) and by our assumption we have

$$\frac{1}{\varepsilon_0^n} \int_{B(\xi_0, R)} \text{tr}(g^{-1}h) [u_0 - |u_0|^{p-2} u_0] w d\mu_g + \frac{1}{\varepsilon_0^{n-2}} \int_M \langle \nabla_g u_0, \nabla_g w \rangle_{b(h)} d\mu_g = 0. \quad (8)$$

Using the normal coordinates at ξ_0 we choose h such that the matrix $\{h_{ij}(x)\}_{i,j=1,\dots,n}$ has the form $h_{12}(x) = h_{21}(x) \in C_0^\infty(B(0, R))$ and $h_{ij} \equiv 0$ otherwise. By (7) we have

$$0 = \int_{B(0, R)} |g(x)|^{1/2} h_{12}(x) \left\{ -\varepsilon_0^2 b_{12}(x) \left(\frac{\partial \tilde{u}_0}{\partial x_1} \frac{\partial \tilde{w}}{\partial x_2} + \frac{\partial \tilde{u}_0}{\partial x_2} \frac{\partial \tilde{w}}{\partial x_1} \right) + \sigma(x) \right\} d\mathfrak{g}$$

where

$$\begin{aligned} \sigma(x) = & -\varepsilon_0^2 \sum_{\substack{r, s = 1, \dots, n \\ (r, s) \neq (1, 2) \\ (r, s) \neq (2, 1)}} b_{rs} \left(\frac{\partial \tilde{u}_0}{\partial x_r} \frac{\partial \tilde{w}}{\partial x_s} \right) \\ & + 2g^{12} \left\{ \frac{\varepsilon_0^2}{2} \sum_{i,j=1}^n g^{ij} \left(\frac{\partial \tilde{u}_0}{\partial x_i} \frac{\partial \tilde{w}}{\partial x_j} \right) + [\tilde{u}_0 - |\tilde{u}_0|^{p-2} \tilde{u}_0] \tilde{w} \right\}. \quad (10) \end{aligned}$$

Here $b_{rs}(x) = (g^{-1}(x)\Gamma g^{-1}(x))_{rs}$, where $\Gamma_{12} = \Gamma_{21} = 0$, $\Gamma_{ij} = \Gamma_{j,i} = 0$ for $i, j = 1, \dots, n$, $(i, j) \neq (1, 2)$. Then $b_{12}(0) = b_{21}(0) = 1$, $b_{rs}(0) = 0$ otherwise, so $\sigma(0) = 0$. By (9), at this point we have

$$-\varepsilon_0^2 b_{12}(x) \left(\frac{\partial \tilde{u}_0}{\partial x_1}(x) \frac{\partial \tilde{w}}{\partial x_2}(x) + \frac{\partial \tilde{u}_0}{\partial x_2}(x) \frac{\partial \tilde{w}}{\partial x_1}(x) \right) + \sigma(x) \text{ for } x \in B(0, R).$$

Then

$$\frac{\partial \tilde{u}_0}{\partial x_1}(0) \frac{\partial \tilde{w}}{\partial x_2}(0) + \frac{\partial \tilde{u}_0}{\partial x_2}(0) \frac{\partial \tilde{w}}{\partial x_1}(0) = 0.$$

If $\xi_0 = \tau \xi_0$, we consider the equality (7) when $h \in \mathcal{S}^k$ vanishes outside $B_g(\xi_0, R)$, recalling that $h(\tau(\xi)) = h(\xi)$ for $\xi \in M$. Arguing as in the previous case, by (9) we get that

$$\gamma(x) = \varepsilon_0^2 b_{12}(x) \left(\frac{\partial \tilde{u}_0}{\partial x_1} \frac{\partial \tilde{w}}{\partial x_2} + \frac{\partial \tilde{u}_0}{\partial x_2} \frac{\partial \tilde{w}}{\partial x_1} \right) + \sigma(x)$$

is antisymmetric with respect to $\bar{\tau} = \exp_{\xi_0}^{-1} \tau \exp_{\xi_0}$. Also, we have that γ is symmetric with respect to $\bar{\tau}$, so $\gamma(0) = 0$, and, since $b_{12}(0) = 1$ and $\sigma(0) = 0$, we have again $\frac{\partial \tilde{u}_0}{\partial x_1}(0) \frac{\partial \tilde{w}}{\partial x_2}(0) + \frac{\partial \tilde{u}_0}{\partial x_2}(0) \frac{\partial \tilde{w}}{\partial x_1}(0) = 0$.

Now we prove that $\frac{\partial \tilde{u}_0(0)}{\partial x_i} \frac{\partial \tilde{w}(0)}{\partial x_i} = 0$ for all $i = 1, \dots, n$.

If $\xi_0 \neq \tau \xi_0$, arguing as in the previous case we get (8). This time we choose the matrix $\{h_{ij}(x)\}_{i,j}$ such that $h_{11} \in C_0^\infty(B(0, R))$, $h_{22} = -h_{11}$ and $h_{ij} \equiv 0$

otherwise. Because $\text{tr}(g^{-1}h) = (g^{11} - g^{22})h_{11}$, by (8), we get

$$\begin{aligned}
0 &= \int_{B(0,R)} |g(x)|^{1/2} h_{11}(x) \left\{ [g^{11}(x) - g^{22}(x)] \times \right. \\
&\quad \times \left(\varepsilon_0^2 \sum_{ij} g^{ij}(x) \frac{\partial \tilde{u}_0}{\partial x_i} \frac{\partial \tilde{w}}{\partial x_j} + \tilde{u}_0 \tilde{w} - |\tilde{u}_0|^{p-2} \tilde{u}_0 \tilde{w} \right) \\
&\quad - \varepsilon_0^2 [g^{11}(x)g^{12}(x) - g^{12}(x)g^{21}(x)] \left(\frac{\partial \tilde{u}_0}{\partial x_1} \frac{\partial \tilde{w}}{\partial x_2} + \frac{\partial \tilde{u}_0}{\partial x_2} \frac{\partial \tilde{w}}{\partial x_1} \right) \\
&\quad \left. - \varepsilon_0^2 \sum_{k=1}^n [(g^{1k}(x))^2 - (g^{2k}(x))^2] \frac{\partial \tilde{u}_0}{\partial x_k} \frac{\partial \tilde{w}}{\partial x_k} \right\} dx. \tag{11}
\end{aligned}$$

Then, recalling that $\frac{\partial \tilde{u}_0}{\partial x_1}(0) \frac{\partial \tilde{w}}{\partial x_2}(0) + \frac{\partial \tilde{u}_0}{\partial x_2}(0) \frac{\partial \tilde{w}}{\partial x_1}(0) = 0$ and that $g^{ij}(0) = \delta_{ij}$ we have

$$\begin{aligned}
&\left[(g^{11}(0))^2 - (g^{21}(0))^2 \right] \frac{\partial \tilde{u}_0(0)}{\partial x_1} \frac{\partial \tilde{w}(0)}{\partial x_1} + \left[(g^{12}(0))^2 - (g^{22}(0))^2 \right] \frac{\partial \tilde{u}_0(0)}{\partial x_2} \frac{\partial \tilde{w}(0)}{\partial x_2} = 0. \\
\text{So } \frac{\partial \tilde{u}_0(0)}{\partial x_1} \frac{\partial \tilde{w}(0)}{\partial x_1} &= \frac{\partial \tilde{u}_0(0)}{\partial x_2} \frac{\partial \tilde{w}(0)}{\partial x_2} \text{ and, analogously, } \frac{\partial \tilde{u}_0(0)}{\partial x_1} \frac{\partial \tilde{w}(0)}{\partial x_1} = \frac{\partial \tilde{u}_0(0)}{\partial x_i} \frac{\partial \tilde{w}(0)}{\partial x_i} \\
&\text{for all } i. \text{ At this point, since } \frac{\partial \tilde{u}_0(0)}{\partial x_i} \frac{\partial \tilde{w}(0)}{\partial x_j} + \frac{\partial \tilde{u}_0(0)}{\partial x_j} \frac{\partial \tilde{w}(0)}{\partial x_i} = 0 \text{ for all } i \neq j \text{ we} \\
&\text{get}
\end{aligned}$$

$$\frac{\partial \tilde{u}_0(0)}{\partial x_i} \frac{\partial \tilde{w}(0)}{\partial x_i} = 0 \text{ for all } i = 1, \dots, n.$$

If $\xi_0 = \tau \xi_0$, since h is symmetric with respect to τ , by (11) we get that

$$\begin{aligned}
\gamma(x) &= [g^{11}(x) - g^{22}(x)] \left(\varepsilon_0^2 \sum_{ij} g^{ij}(x) \frac{\partial \tilde{u}_0}{\partial x_i} \frac{\partial \tilde{w}}{\partial x_j} + \tilde{u}_0 \tilde{w} - |\tilde{u}_0|^{p-2} \tilde{u}_0 \tilde{w} \right) \\
&\quad - \varepsilon_0^2 [g^{11}(x)g^{12}(x) - g^{12}(x)g^{21}(x)] \left(\frac{\partial \tilde{u}_0}{\partial x_1} \frac{\partial \tilde{w}}{\partial x_2} + \frac{\partial \tilde{u}_0}{\partial x_2} \frac{\partial \tilde{w}}{\partial x_1} \right) \\
&\quad - \varepsilon_0^2 \sum_{k=1}^n [(g^{1k}(x))^2 - (g^{2k}(x))^2] \frac{\partial \tilde{u}_0}{\partial x_k} \frac{\partial \tilde{w}}{\partial x_k}
\end{aligned}$$

is antisymmetric with respect to $\bar{\tau} = \exp_{\xi_0}^{-1} \tau \exp_{\xi_0}$. Concluding

$$0 = \gamma(0) = (g^{11}(0))^2 \frac{\partial \tilde{u}_0(0)}{\partial x_1} \frac{\partial \tilde{w}(0)}{\partial x_1} - (g^{22}(0))^2 \frac{\partial \tilde{u}_0(0)}{\partial x_2} \frac{\partial \tilde{w}(0)}{\partial x_2}.$$

At this point, arguing as above we have that

$$\frac{\partial \tilde{u}_0(0)}{\partial x_i} \frac{\partial \tilde{w}(0)}{\partial x_i} = 0 \text{ for all } i = 1, \dots, n.$$

and the Step 2 is proved.

Step 3. Conclusion of the proof.

By Step 2, we have that, for any $h \in \mathcal{S}^k$

$$0 = \langle F'_{\varepsilon, h}(\varepsilon_0, h_0, u_0) [0, h], w \rangle_{H^1_{\varepsilon_0, g_0 + h_0}} = \frac{1}{2\varepsilon_0^n} \int_M \text{tr}(g^{-1}h) u_0 (1 - |u_0|^{p-2}) w d\mu_g. \quad (12)$$

Here $g = g_0 + h_0$. Moreover it holds

$$-\varepsilon_0 \Delta_g w + w = (p-1)|u_0|^{p-2} w \quad w \in H_g^\tau \quad (13)$$

We choose $h(\xi) = \alpha(\xi)g(\xi)$ for any $\alpha \in C^\infty(M)$ with $\alpha(\tau\xi) = \alpha(\xi)$, so, by (12), the function $u_0 (1 - |u_0|^{p-2}) w$ is antisymmetric with respect to the involution τ . Furthermore $u_0 (1 - |u_0|^{p-2}) w$ is also symmetric, so

$$u_0 (1 - |u_0|^{p-2}) w \equiv 0. \quad (14)$$

By contradiction we assume that w does not vanish indentially in M . Since $u_0 \in H_g^\tau \setminus \{0\}$ we can split

$$M = M^0 \cup M^1 \cup \tau M^1 \cup M^+ \cup \tau M^+$$

where $M^0 = \{x \in M : u_0(x) = 0\}$, $M^1 = \{x \in M : u_0(x) = 1\}$, and $M^+ = \{x \in M : u_0(x) > 0, u_0(x) \neq 1\}$. By (14) we have that $w \equiv 0$ on the open subset $M^+ \cup \tau M^+$. Also, we notice that M_0 and M_1 are disjoint sets because u_0 is a continuous function. By this, and by (13), we have that $-\varepsilon_0 \Delta_g w + w = 0$ on M_0 and $w = 0$ on ∂M_0 . By the maximum principle, we conclude that $w = 0$ on M_0 . So we have that, by (13), $-\varepsilon_0 \Delta_g w + w = (p-1)w$ on the whole M . On the other hand, by [1], we have that $\mu_g(\{x \in M : w(x) = 0\}) = 0$. A contradiction arises and that concludes the proof \square

With the same argument we can prove the following corollary.

Corollary 13. *Given ε_0 , for any $(h_0, u_0) \in \mathcal{B}_\rho \times H_{g_0}^\tau \setminus \{0\}$ such that $F(\varepsilon_0, h_0, u_0) = 0$, if $w \in \ker F'_u(\varepsilon_0, h_0, u_0)$ and*

$$\langle F'_h(\varepsilon_0, h_0, u_0) [h], w \rangle_{H^1_{\varepsilon_0, g_0 + h_0}} = 0 \quad \forall h \in \mathcal{S}^k,$$

then $w = 0$.

Lemma 14. *Given $g_0 \in \mathcal{M}^k$ and ε_0 , if there exists a number $\mu > m_{\varepsilon_0, g_0}$ not a critical level of the functional J_{ε_0, g_0} , then, for ρ small enough, the map $\pi \circ i : G^{-1}(0) \rightarrow \mathcal{S}^k$ is proper. Here G is defined in (3), i is the canonical embedding from $G^{-1}(0)$ into $\mathcal{S}^k \times H_{g_0}^\tau$ and π is the projection from $\mathcal{S}^k \times H_{g_0}^\tau$ into \mathcal{S}^k .*

Proof. Let $\{u_n\} \subset \mathcal{D}$, where

$$\mathcal{D} = \{u \in H_{g_0}^\tau : \mu_0 < J_{\varepsilon_0, g_0}(u) < \mu\},$$

and μ_0 is an arbitrary number in $(0, m_{\varepsilon_0, g_0}^\tau)$. It is sufficient to prove that if u_n satisfies $-\varepsilon_0^2 \Delta_{g_0 + h_n} u_n + u_n = |u_n|^{p-2} u_n$ with $h_n \rightarrow h_0 \in \mathcal{B}_\rho$, then the sequence $\{u_n\}$ has a subsequence convergent in \mathcal{D} . First we show that $\{u_n\}$ is bounded in $H_{g_0}^\tau$. Since the sets $H_{g_0+h}^1(M)$ and $H_{g_0}^1(M)$ are the same in $h \in \mathcal{B}_\rho$

and the norms $\|\cdot\|_{H_{g_0+h}^1}$ and $\|\cdot\|_{H_{g_0}^1}$ are equivalent with equivalence constants c_1 and c_2 not depending on h , we have

$$c_1 \|u\|_{H_{\varepsilon_0, g_0}^1} \leq \|u\|_{H_{\varepsilon_0, g_0+h}^1} \leq c_2 \|u\|_{H_{\varepsilon_0, g_0}^1}.$$

By this, and because $u_n \in \mathcal{N}_{\varepsilon_0, g_0+h_n}^\tau$ we have

$$\begin{aligned} \left(\frac{1}{2} - \frac{1}{p}\right) c_1^2 \|u_n\|_{H_{\varepsilon_0, g_0}^1}^2 &\leq \left(\frac{1}{2} - \frac{1}{p}\right) \|u_n\|_{H_{\varepsilon_0, g_0+h_n}^1}^2 = J_{\varepsilon_0, g_0+h_n}(u_n) = \\ &= \frac{1}{2} \|u_n\|_{H_{\varepsilon_0, g_0+h_n}^1}^2 - \frac{1}{p} \|u_n\|_{L_{\varepsilon_0, g_0+h_n}^p}^p \\ &\leq J_{\varepsilon_0, g_0}(u_n) + c \|h_n\|_k \left[\|u_n\|_{H_{\varepsilon_0, g_0}^1}^2 + \|u_n\|_{L_{\varepsilon_0, g_0}^p}^p \right] \leq \\ &\leq \mu + c\rho \left[\|u_n\|_{H_{\varepsilon_0, g_0}^1}^2 + \|u_n\|_{L_{\varepsilon_0, g_0}^p}^p \right] \end{aligned} \quad (15)$$

Moreover, since $\mu_0 < J_{\varepsilon_0, g_0}(u_n) < \mu$ we get

$$\mu_0 < \left(\frac{1}{2} - \frac{1}{p}\right) \|u_n\|_{L_{\varepsilon_0, g_0}^p}^p < \mu \quad (16)$$

by (15) and (16), if $\|u_n\|_{H_{\varepsilon_0, g_0}^1} \rightarrow +\infty$ we get

$$\left(\frac{1}{2} - \frac{1}{p}\right) \|u_n\|_{H_{\varepsilon_0, g_0}^1}^2 \leq \mu + c\rho \|u_n\|_{H_{\varepsilon_0, g_0}^1}^2 + c\rho,$$

then, choosing ρ small enough, we get the contradiction.

Since the sequence $\{u_n\}$ is bounded in $H_{g_0}^\tau$ and $H_{g_0+h_0}^\tau$, up to a subsequence $u_n \rightarrow u$ in $L_{g_0+h_0}^{t, \tau}(M)$ and $L_{g_0}^{t, \tau}(M)$ for $2 \leq t < 2^*$. Then

$$i_{\varepsilon_0, g_0+h_0}^* (|u_n|^{p-2} u_n) = A_{h_0}^{\varepsilon_0} (|u_n|^{p-2} u_n) \rightarrow A_{h_0}^{\varepsilon_0} (|u|^{p-2} u) \text{ in } H_{\varepsilon_0, g_0+h_0}^\tau \quad (17)$$

Since the map A is of class C^1 (see Lemma 7) we have, for some $\theta \in (0, 1)$

$$\begin{aligned} &\|A_{h_n}^{\varepsilon_0} (|u_n|^{p-2} u_n) - A_{h_0}^{\varepsilon_0} (|u_n|^{p-2} u_n)\|_{H_{\varepsilon_0, g_0}^1} \\ &= \|A'(\varepsilon_0, h_0 + \theta(h_n - h_0)) [0, h_n - h_0] (|u_n|^{p-2} u_n)\|_{H_{\varepsilon_0, g_0}^1} \\ &\leq \| |u_n|^{p-1} \|_{L_{\varepsilon_0, g_0}^{p'}} \|h_n - h_0\|_k \|A'(\varepsilon_0, h_0 + \theta(h_n - h_0))\|_{\mathcal{L}(\mathcal{B}_\rho, \mathcal{L}(L_{\varepsilon_0, g_0}^{p', \tau}, H_{\varepsilon_0, g_0}^\tau))} \end{aligned} \quad (18)$$

By (17) and (18) we get $A_{h_n}^{\varepsilon_0} (|u_n|^{p-2} u_n) \rightarrow A_{h_0}^{\varepsilon_0} (|u|^{p-2} u)$ in $H_{\varepsilon_0, g_0}^\tau$.

Since $0 = u_n - A_{h_n}^{\varepsilon_0} (|u_n|^{p-2} u_n)$ we get that u_n converges to u in $H_{g_0}^\tau$. Moreover $u - A_{h_0}^{\varepsilon_0} (|u|^{p-2} u) = 0$. Since μ_0 and μ are not critical values for $J_{\varepsilon_0, g_0}(u)$, we have that $\mu_0 < J_{\varepsilon_0, g_0}(u) < \mu$. Then $u \in \mathcal{D}$. \square

5 An application

In this section we choose $\tau = -\text{Id}$ and the manifold M invariant with respect to the involution $\tau = -\text{Id}$. We also assume $0 \notin M$, so $M_\tau = \emptyset$. Using the previous results of genericity for non degenerate sign changing solutions of problem (2) we obtain a lower bound on the number of non degenerate solutions which change

sign exactly once. This estimate is formulated also in [17]. In the cited paper this result is proved under an assumption on non degeneracy of critical points that we do not need.

We sketch the proof of propositions 4 and 5 showing how we use the results of genericity for non degeneracy of critical points to obtain the same estimate.

We recall that there exists a unique positive spherically symmetric function $U \in H^1(\mathbb{R}^n)$ such that $-\Delta U + U = U^{p-1}$ in \mathbb{R}^n . Also, it is well known that for any $\varepsilon > 0$, $U_\varepsilon(x) := U\left(\frac{x}{\varepsilon}\right)$ is a solution of $-\varepsilon^2 \Delta U_\varepsilon + U_\varepsilon = U_\varepsilon^{p-1}$ in \mathbb{R}^n .

Let g_0 be in \mathcal{M}^k and h be in \mathcal{B}_ρ for some $\rho > 0$. Let us define a smooth cut off real function χ_R such that $\chi_R(t) = 1$ if $0 \leq t \leq R/2$, $\chi_R(t) = 0$ if $t \geq R$ and $|\chi'(t)| < 2/R$. Fixed $q \in M$ and $\varepsilon > 0$ we define on M the function

$$W_{q,\varepsilon}^g(x) = \begin{cases} U_\varepsilon(\exp_q^{-1}(x))\chi_R(|\exp_q^{-1}(x)|) & \text{if } x \in B_g(q, R) \\ 0 & \text{otherwise} \end{cases},$$

where $B_g(q, R)$ is the geodesic ball of radius R centered at q . We choose R smaller than the injectivity radius of M and such that $B_g(q, R) \cap B_g(-q, R) = \emptyset$. Here and in the following we set $g = g_0 + h$.

We can define a map $\Phi_{\varepsilon,g} : M \rightarrow \mathcal{N}_{\varepsilon,g}^\tau$ as

$$\Phi_{\varepsilon,g}(q) = t(W_{q,\varepsilon}^g) W_{q,\varepsilon}^g - t(W_{-q,\varepsilon}^g) W_{-q,\varepsilon}^g.$$

Here

$$[t(W_{q,\varepsilon}^g)]^{p-2} = \frac{\int_M \varepsilon^2 |\nabla_g W_{q,\varepsilon}^g|^2 + |W_{q,\varepsilon}^g|^2 d\mu_g}{\int |W_{q,\varepsilon}^g|^p d\mu_g},$$

thus $t(W_{q,\varepsilon}^g) W_{q,\varepsilon}^g \in \mathcal{N}_{\varepsilon,g}$ and we have $\Phi_{\varepsilon,g}(q) = -\Phi_{\varepsilon,g}(-q)$. Now we can define

$$\begin{aligned} \tilde{\Phi}_{\varepsilon,g} & : M/G \rightarrow \mathcal{N}_{\varepsilon,g}^\tau / \mathbb{Z}_2 \\ \tilde{\Phi}_{\varepsilon,g}[q] & = [\Phi_{\varepsilon,g}(q)] = \{\Phi_{\varepsilon,g}(q), \Phi_{\varepsilon,g}(-q)\} \end{aligned}$$

where

$$M/G = \{[q] = (q, -q) : q \in M\} \quad \mathcal{N}_{\varepsilon,g}^\tau / \mathbb{Z}_2 = \{(u, -u) : u \in \mathcal{N}_{\varepsilon,g}^\tau\}.$$

We set $\tilde{J}_{\varepsilon,g}[u] = J_{\varepsilon,g}(u)$. Obviously, $\tilde{J}_{\varepsilon,g} : \mathcal{N}_{\varepsilon,g}^\tau / \mathbb{Z}_2 \rightarrow \mathbb{R}$.

Lemma 15. *For any $\delta > 0$ there exists $\varepsilon_2 = \varepsilon_2(\delta)$ such that, if $\varepsilon < \varepsilon_2$ then*

$$\tilde{\Phi}_{\varepsilon,g_0+h}([q]) \in \mathcal{N}_{\varepsilon,g_0+h}^\tau \cap \tilde{J}_{\varepsilon,g_0+h}^{2(m_\infty+\delta)} \quad \forall h \in \mathcal{B}_\rho.$$

Moreover we have that

$$\lim_{\varepsilon \rightarrow 0} m_{\varepsilon,g_0+h}^\tau = 2m_\infty \text{ uniformly on } h \in \mathcal{B}_\rho.$$

For a proof of this result we refer to [3].

For any function $u \in \mathcal{N}_{\varepsilon,g_0+h}^\tau$ we define

$$\beta_g(u) = \frac{\int_M x(u^+)^p d\mu_g}{\int_M (u^+)^p d\mu_g}$$

where $g = g_0 + h$. Also, we define

$$\begin{aligned}\tilde{\beta}_g &: (\mathcal{N}_{\varepsilon, g_0+h}^\tau / \mathbb{Z}_2) \cap \tilde{J}_{\varepsilon, g_0+h}^{2(m_\infty+\delta)} \rightarrow M_d/G \\ \tilde{\beta}_g([u]) &:= [\tilde{\beta}_g(u)] = \{\beta_g(u), \beta_g(-u)\} = \{\beta_g(u), -\beta_g(u)\}\end{aligned}$$

where $M_d = \{u \in M : d(x, M) < d\}$.

Lemma 16. *There exists $\tilde{\delta}$ such that $\forall \delta < \tilde{\delta}$ there exists $\tilde{\varepsilon} = \tilde{\varepsilon}(\delta)$ and for any $\varepsilon < \tilde{\varepsilon}$ the map*

$$\tilde{\beta}_g \circ \tilde{\Phi}_{\varepsilon, g} : M/G \xrightarrow{\tilde{\Phi}_{\varepsilon, g}} (\mathcal{N}_{\varepsilon, g_0+h}^\tau / \mathbb{Z}_2) \cap \tilde{J}_{\varepsilon, g_0+h}^{2(m_\infty+\delta)} \xrightarrow{\tilde{\beta}_g} M_d/G$$

is continuous and homotopic to identity, for all $g = g_0 + h$ with $h \in \mathcal{B}_\rho$.

For a proof of this result we refer to [3].

Let us sketch the proof of Proposition 4. We are going to find an estimate on the number of pairs non degenerate critical points $(u, -u)$ for the functional $J_{\varepsilon, g} : H_g^\tau \rightarrow \mathbb{R}$ with energy close to $2m_\infty$ with respect to the parameters $(\varepsilon, h) \in (0, \tilde{\varepsilon}) \times \mathcal{B}_\rho$ for $\tilde{\varepsilon}, \rho$ small enough.

We recall that, by Theorem 1, given the positive numbers $\tilde{\varepsilon}, \rho$, the set

$$D(\tilde{\varepsilon}, \rho) = \left\{ (\varepsilon, h) \in (0, \tilde{\varepsilon}) \times \mathcal{B}_\rho : \begin{array}{l} \text{any } u \in H_{g_0}^\tau \text{ solution of} \\ -\varepsilon^2 \Delta_{g_0+h} u + u = |u|^{p-2} u \text{ is not degenerate} \end{array} \right\}$$

is a residual subset in $(0, \tilde{\varepsilon}) \times \mathcal{B}_\rho$. Since

$$\lim_{(\varepsilon, h) \rightarrow 0} m_{\varepsilon, g_0+h}^\tau = 2m_\infty,$$

given $\delta \in (0, \frac{m_\infty}{4})$, for (ε, h) small enough we have

$$0 < 2(m_\infty - \delta) < m_{\varepsilon, g_0+h}^\tau < 2(m_\infty + \delta) < 3m_\infty,$$

thus $2(m_\infty - \delta)$ is not a critical value of $J_{\varepsilon, g}$ on H_g^τ . At this point we take $(\varepsilon, h) \in D(\tilde{\varepsilon}, \rho)$ with $\tilde{\varepsilon}, \rho$ small enough. Thus the critical points of $J_{\varepsilon, g}$ such that $J_{\varepsilon, g} < 3m_\infty$ are in a finite number by Theorem 1, and then we can assume that $2(m_\infty + \delta)$ is not a critical value of $J_{\varepsilon, g}$.

Let $\mathcal{N}_\varepsilon^\tau / \mathbb{Z}_2$ be the set obtained by identifying antipodal points of the Nehari manifold $\mathcal{N}_\varepsilon^\tau$. It is easy to check that the set $\mathcal{N}_\varepsilon^\tau / \mathbb{Z}_2$ is homeomorphic to the projective space $\mathbb{P}^\infty = \Sigma / \mathbb{Z}_2$ obtained by identifying antipodal points in Σ , Σ being the unit sphere in H_g^τ . We are looking for pairs of nontrivial critical points $(u, -u)$ of the functional $J_\varepsilon : H_g^\tau \rightarrow \mathbb{R}$, that is searching for critical points of the functional

$$\begin{aligned}\tilde{J}_{\varepsilon, g} &: (H_g^\tau \setminus \{0\}) / \mathbb{Z}_2 \rightarrow \mathbb{R}; \\ \tilde{J}_{\varepsilon, g}([u]) &:= J_{\varepsilon, g}(u) = J_{\varepsilon, g}(-u).\end{aligned}$$

We use the same Morse theory argument as in [4]. The following result can be found in ([2] and Lemma 5.2 of [4])

$$P_t \left(\tilde{J}_{\varepsilon, g}^{2(m_\infty+\delta)}, \tilde{J}_{\varepsilon, g}^{2(m_\infty-\delta)} \right) = t P_t \left(\tilde{J}_{\varepsilon, g}^{2(m_\infty+\delta)} \cap (\mathcal{N}_\varepsilon^\tau / \mathbb{Z}_2) \right). \quad (19)$$

By Lemma 15 and Lemma 16 we have that $\tilde{\beta}_g \circ \tilde{\Phi}_{\varepsilon,g} : M/G \rightarrow M_d/G$ is a map homotopic to the identity of M/G and that M_d/G is homotopic to M/G . Therefore we have

$$P_t \left(\tilde{J}_{\varepsilon,g}^{2(m_\infty+\delta)} \cap (\mathcal{N}_\varepsilon^\tau / \mathbb{Z}_2) \right) = P_t(M/G) + Z(t) \quad (20)$$

were $Z(t)$ is a polynomial with non negative coefficients. Since the functional $J_{\varepsilon,g}$ satisfies the Palais Smale condition by the compactness of M , and the critical points of $J_{\varepsilon,g}$ in $J_{\varepsilon,g}^{3m_\infty}$ are non degenerate (because $(\varepsilon, h) \in D(\tilde{\varepsilon}, \rho)$), by Morse Theory and relations (19) and (20) we get at least $P_1(M/G)$ pairs $(u, -u)$ of non trivial solutions of $-\varepsilon^2 \Delta_g u + u = |u|^{p-2}u$ with $J_{\varepsilon,g}(u) = J_{\varepsilon,g}(-u) < 3m_\infty$. So, these solutions change sign exactly once. That concludes the proof of Proposition 4.

Remark 17. In the same way we obtain that, given $g_0 \in \mathcal{M}^k$ and $\varepsilon_0 > 0$, the set

$$\mathcal{A}^* = \left\{ \begin{array}{l} h \in \mathcal{B}_\rho : \text{ the equation } -\varepsilon_0^2 \Delta_{g_0+h} u + u = |u|^{p-2}u \\ \text{ has at least } P_1(M/G) \text{ pairs of non degenerate solutions} \\ (u, -u) \in H_g^\tau \setminus \{0\} \text{ which change sign exactly once} \end{array} \right\}$$

is a residual subset of \mathcal{B}_ρ .

The proof of Proposition 5 can be obtained with similar arguments.

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